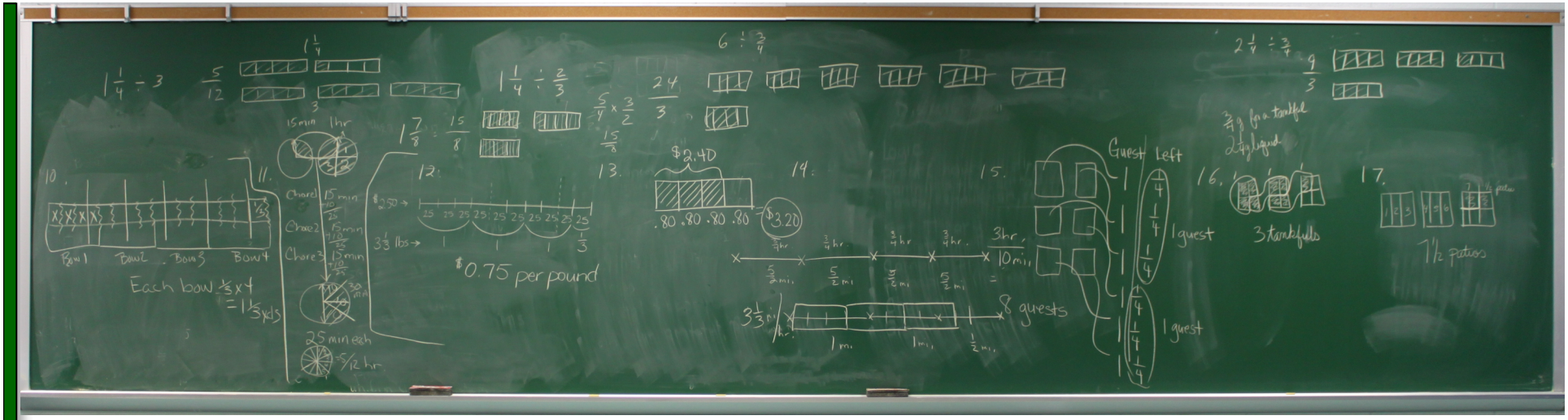


Effective Fractions Instruction:

Recommendations from a What Works Clearinghouse Practice Guide

2010 NCTM Regional Conference & Exposition

October 15, 2010 Baltimore, Maryland

Dr. Francis (Skip) Fennell
Professor of Education, McDaniel College
Past President, National Council of Teachers of Mathematics

Jon Wray
Facilitator, Howard County Public Schools
Past President, Maryland Council of Teachers of Mathematics

Panel

Robert Siegler (Chair)
Carnegie Mellon University

Thomas Carpenter
University of Wisconsin-Madison

Francis (Skip) Fennell
McDaniel College

David Geary
University of Missouri at Columbia

James Lewis
University of Nebraska-Lincoln

Yukari Okamoto
University of California-Santa Barbara

Laurie Thompson
Elementary Teacher

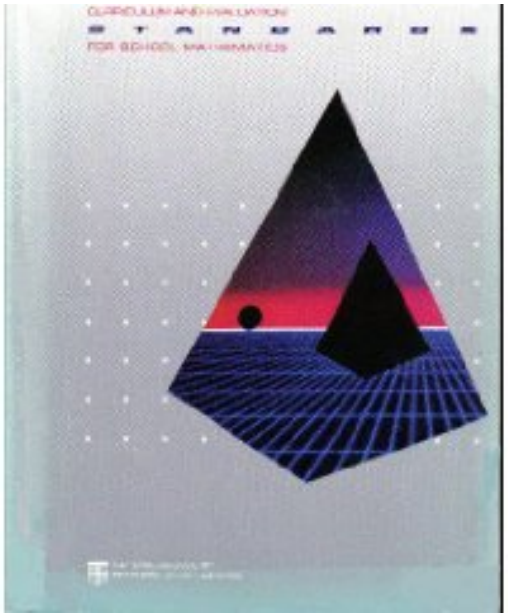
Jonathan (Jon) Wray
Howard County (MD) Public Schools

Staff

Jeffrey Max
Andrew Gothro
Sarah Prenovitz
Mathematical Policy Research

Project Officer - **Susan Sanchez**
Institution of Education Sciences (IES)

Number Sense - History



- Number Meaning
- Relationships
- Magnitude
- Operation Sense
- Real Life Number Sense - Applications

Howden, 1989

Making Sense of Numbers…

1. Ability to compose and decompose numbers…
2. Ability to recognize the relative magnitude of numbers – including comparing and ordering.
3. Ability to deal with the absolute magnitude of numbers – realizing, for instance there are far fewer than 500 people in this session!
4. Ability to use benchmarks.
5. Ability to link numeration, operation, and relation symbols in meaningful ways.
6. Understanding the effects of operations on numbers.
7. The ability to perform mental computation through invented strategies that take advantages of numerical and operational properties.
8. Being able to use numbers flexibly to estimate numerical answers to computations, and to recognize when an estimate is appropriate.
9. A disposition towards making sense of numbers.

"It is possible to have good number sense for whole numbers, but not for fractions…"

Sowder, J. and Schappelle, Eds., 1989

How did this get started…

Fraction issues…

- Conceptual Knowledge and Skills
- Learning Processes
- Assessment
- Survey of Algebra Teachers

How did this get started…
NMAP - Student Preparation

The first question concerned the adequacy of student preparation coming into the Algebra I classes. The topics that were rated as especially problematic were:

- Rational numbers;
- Solving word problems, and;
- Basic study skills.

Final Report on the National Survey of Algebra Teachers for the National Math Panel, NORC, September, 2007

American students' weak understanding of fractions

- 2004 NAEP - 50% of 8th-graders could not order three fractions from least to greatest (NCTM, 2007)
- 2004 NAEP, Fewer than 30% of 17-year-lds correctly translated 0.029 as 29/1000 (Kloosterman, 2010)
- One-on-one controlled experiment tests - when asked which of two decimals, 0.274 and 0.83 is greater, most 5th- and 6th-graders choose 0.274 (Rittle-Johnson, Siegler, and Alibali, 2001)
- Knowledge of fractions differs even more between students in the U.S. and students in East Asia than des knowledge of whole numbers (Mullis, et al., 1997)

Facets of the lack of student conceptual understanding… just a few

- Not viewing fractions as numbers at all, but rather as meaningless symbols that need to be manipulated in arbitrary ways to produce answers that satisfy a teacher
- Focusing on numerators and denominators as separate numbers rather than thinking of the fraction as a single number.
- Confusing properties of fractions with those of whole numbers

IES – Practice Guide - Fractions, 2010 (pp. 6-7)

Fractions Guide authors concluded:

"A high percentage of U.S. students lack conceptual understanding of fractions, even after studying fractions for several years; this, in turn, limits students' ability to solve problems with fractions and to learn and apply computational procedures involving fractions."

IES – Practice Guide - Fractions, 2010 (pp. 6-7)

Common Core State Standards

Grade 1 – Geometry

- Partition circles and rectangles into two and four equal shares, describe halves, fourths,…

Grade 2 - Geometry

- Partition circles and rectangles into two, three, or four equal shares, describe halves, thirds, fourths, …. Describe the whole as two halves, three thirds, four fourths. Recognize that equal shares need not have the same shape.

Common Core State Standards

Grade 3 – N&O Fractions

- Develop understanding of fractions as numbers.

Grade 4 – N&O Fractions

- Extend understanding of fraction equivalence and ordering.
- Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
- Understand decimal notation for fractions and compare fractions.

Grade 5 – N&O Fractions

- Use equivalent fractions as a strategy to add and subtract fractions.
- Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

Common Core State Standards

Grade 6 – Ratios and Proportional Reasoning

- Understand ratio concepts and use ratio reasoning to solve problems.

Grade 6 – The Number System

- Apply and extend previous understandings of multiplication and division to divide fractions by fractions
- Apply and extend previous understandings of numbers to the system of rational numbers.

Grade 7 – Ratios and Proportional Reasoning

- Analyze proportional relationships and use them to solve real-world and mathematical problems.

Grade 7 – The Number System

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

Focus and Coherence - CCSS

- Informal Beginnings
  - Grades 1, 2
- **Number and Operations – Fractions**
  - **Grades 3-5**
- **Ratios and Proportional Reasoning**
  - **Grades 6, 7**
- The Number System
  - Grades 6, 7

What about research?

"…has a terrible time with basic skills. I mean, if we ever do anything with fractions, she's lost."*

*a lament by and about far too many…

Usher, …Middle School Students' Self-Efficacy in Mathematics, AERJ, 2009

2nd Handbook Research on Mathematics Teaching & Learning

- Whole Number Concepts and Operations
  - Citations: 334
- Rational Numbers and Proportional Reasoning
  - Citations: 140
- In the 2000's: only 9 citations;
  - 109 in Whole Number Concepts and Operations
  - 1/12th

- "The number of references in this chapter predating 1992 is far greater than the number appearing since the last handbook."

- "This crisis…stems from:
  - Teachers are not prepared to teach content other than part-whole fractions;
  - Long-term commitment is needed because rational number topics are learned over many years.
  - The nonlinear development of the content does not mesh well with scope and sequence currently prescribing mathematics instruction in schools; and
  - In comparison to a domain such as early addition and subtraction, little research progress is evident."

Lamon, 2007 in 2nd Handbook Research on Mathematics Teaching & Learning

JRME and Fractions…

- March 2005
  - Representing Fractions with Standard Notation: A Developmental Analysis. Geoffrey B. Saxe, Edd V. Taylor, Clifton McIntosh and Maryl Gearhart
- January 2008
  - Teaching and Learning Fraction Addition on Number Lines: Andrew Izsák, Erik Tillema and Zelha Tunç-Pekkan
- May 2008
  - Learning to Use Fractions: Examining Middle School Students' Emerging Fraction Literacy: Debra I. Johanning
- July 2008
  - Josh's Operational Conjectures: Abductions of a Splitting Operation and the Construction of New Fractional Schemes: Anderson Norton
- March 2009
  - Students' Overuse of Proportionality on Missing-Value Problems: How Numbers May Change Solutions: Wim Van Dooren, Dirk De Bock, Marleen Evers and Lieven Verschaffel

5 manuscripts in approx 4+ years

Another look…

NAEP, 2007 – Public Release Items

Grade 4

4. What fraction of the group of umbrellas is closed?

A $\frac{1}{3}$

B $\frac{3}{7}$

C $\frac{4}{7}$

D $\frac{3}{4}$

Correct 80%
Incorrect 19%
Omit 1%

NAEP, 2007

Grade 4

What fraction of the figure is shaded?

Correct: 77%
Incorrect: 22%
Omit: 1%

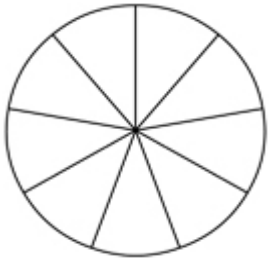
Answer: _____

Did you use the calculator on this question?

NAEP, 2007

Grade 4

Luis wants to make a game spinner in which the chance of landing on blue will be twice the chance of landing on red. He is going to label each section either red (R) or blue (B). Show how he could label his spinner.



11% satisfactory
8% extended
14% partial
4% minimal
59% incorrect

Number of blues _____

Number of reds _____

NAEP, 2007

Grade: 8

- In which of the following are the three fractions arranged from least to greatest?

A) $\frac{2}{7}, \frac{1}{2}, \frac{5}{9}$

B) $\frac{1}{2}, \frac{2}{7}, \frac{5}{9}$

C) $\frac{1}{2}, \frac{5}{9}, \frac{2}{7}$

D) $\frac{5}{9}, \frac{1}{2}, \frac{2}{7}$

E) $\frac{5}{9}, \frac{2}{7}, \frac{1}{2}$

49% correct
49% incorrect
1% omit

○ Yes ○ No

Did you use the calculator on this question?

NAEP, 2007

Grade 8

Tammy scored 52 out of 57 possible points on a quiz. Which of the following is closest to the percent of the total number of points that Tammy scored?

A) 0.91%
B) 1.10%
C) 52%
D) 91%
E) 95%

62% correct
37% incorrect
1% omit

Did you use the calculator on this question?

NAEP, 2007

Grade 8

The temperature in degrees Celsius can be found by subtracting 32 from the temperature in degrees Fahrenheit and multiplying the result by 5/9. If the temperature of a furnace is 393 degrees Fahrenheit, what is it in degrees Celsius, to the nearest degree?

A) 649
B) 375
C) 219
D) 201
E) 187

35% correct
61% incorrect
3% omit

Did you use the calculator on this question?

NAEP, 2007

Hmm

- “In my school, teachers can choose from fraction bars, fraction circles, fraction pieces that snap together and other concrete materials.”
- “I feel my rationale for choosing a manipulative for instruction is selfish – it should not be about me, it should be about the kids.”
- “What fraction representation would best enhance student achievement at the 5th grade level?”

Erin Prajzner, April 12, 2010

Think about…
Curriculum, Assessments, Research…

IES* Practice Guide

- The research base for the guide was identified through a comprehensive search for studies over the past 20 years that evaluated teaching and learning about fractions.
- The process yielded more than 3,000 citations. Of these, 132 met the What Works Clearinghouse criteria for review (4%), and 33 met the causal validity standards of the WWC (1+%).

*Institute of Education Sciences

What's in a "Practice Guide?"

- Audience and grade level (p. 7)
- Review of recommendations (p. 1)
- Levels of evidence (p. 3)
- Introduction (p. 6)
- Recommendations, "Roadblocks" (e.g., p. 18) and Suggestions (pp. 12-46)
  - Example: **Roadblock 1.3** (p. 18) - *When creating equal shares, students do not distinguish between the number of things shared and the quantity shared (confusing equal numbers of shares with equal amounts of shared).* **Suggested Approach**…
- Glossary (p. 47)
- Appendices, endnotes, index, references, etc.(pp. 49-84)

Levels of Evidence

- **Strong** - positive findings are demonstrated in multiple well-designed, well-executed studies, leaving little or no double that the positive effects are caused by the recommended practice
- **Moderate** - well-designed studies show positive impacts, but some questions remain about whether the findings can be generalized or whether the studies definitively show that the practice is effective.
- **Minimal** - data may; suggest a relationship between the recommended practice and positive outcomes, but research has not demonstrated that the practice is the cause of positive outcomes.

IES – Practice Guide - Fractions, 2010 (See Table 1 on pp. 4-5)

Recommendations

1. Build on students' **informal understanding of sharing and proportionality** to develop initial fraction concepts. **(Minimal)**
2. Help students recognize that **fractions are numbers and that they expand the number system beyond whole numbers**. Use **number lines as a central representational tool** in teaching this and other fraction concepts from the early grades onward. **(Moderate)**
3. Help students **understand why procedures for computations** with fractions **makes sense**. **(Moderate)**
4. Develop students' **conceptual understanding** of strategies for solving **ratio, rate, and proportion problems before exposing them to cross-multiplication** as a procedure to use to solve such problems. **(Minimal)**
5. Professional development programs should place a high priority on improving teachers' **understanding of fractions** and **how to teach them**. **(Minimal)**

IES – Practice Guide - Fractions, 2010

Recommendation 1

Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts.

- Use equal-sharing activities to introduce the concept of fractions. Use sharing activities that involve dividing sets of objects as well as single whole objects.
- Extend equal-sharing activities to develop students' understanding of ordering and equivalence of fractions.
- Build on students' informal understanding to develop more advanced understanding of proportional reasoning concepts. Begin with activities that involve similar proportions, and progress to activities that involve ordering different proportions.

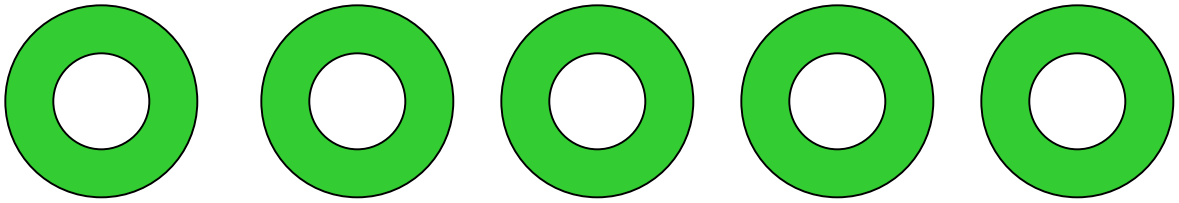
Introducing fractions using sharing

- Four children want to share 10 cookies. Each child gets the same amount. How many cookies does each child get?
- Why?
  - Sharing is intuitive
  - Solution combines wholes and fractions
  - Sharing and repeated halving

- How can we share **eleven** hoagies (aka subs) among four people?

- How can we share **eleven** hoagies (aka subs) among five people?

“How can I show 6 of 5, when I only have 5?”



Sometimes it’s language…

How about if we have six people and we need to share 5 cookies?*

Division involving equal shares is a process that many understand intuitively.

*food seems to work – a lot!

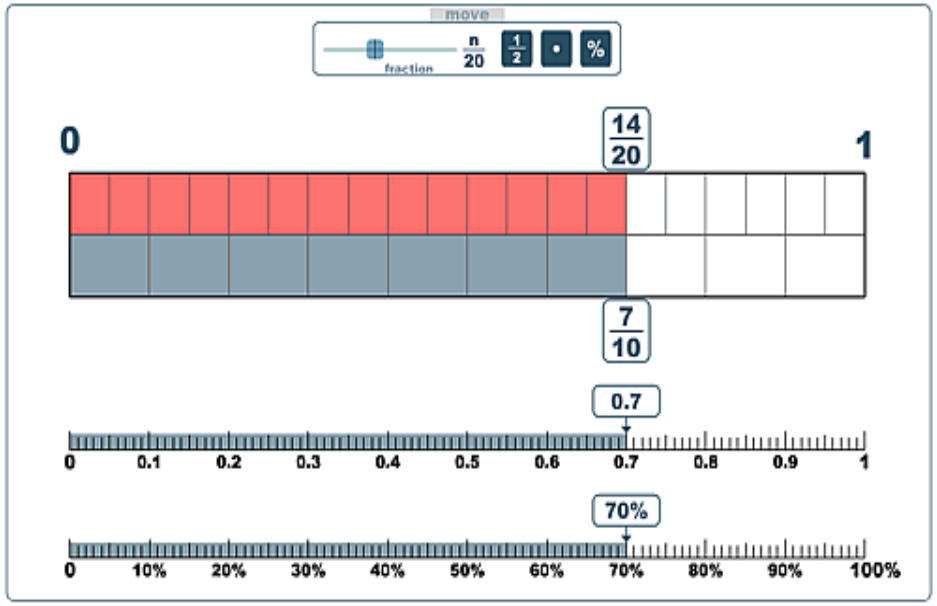
Recommendation 2

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

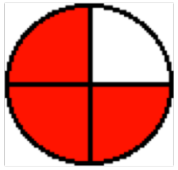
- Use measurement activities and number lines to help students understand that fractions are numbers, with all the properties that numbers share.
- Provide opportunities for students to locate and compare fractions on number lines.
- Use number lines to improve students' understanding of fraction equivalence, fraction density (the concept that there are an infinite number of fractions between any two fractions), and negative fractions.
- Help students understand that fractions can be represented as common fractions, decimals, and percentages, and develop students' ability to translate among these forms.

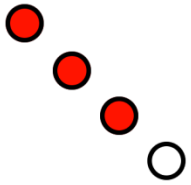
Name…

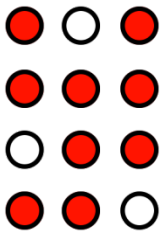
- A fraction between:
  - 0 and 1
  - ¼ and ½
  - 1/3 and 2/3
  - 5/6 and 1
  - 0 and 1/8
- A decimal between:
  - 1.1 and 1.2
  - 1.10 and 1.11
- …



Thinking about ¾…

a) 

b) 

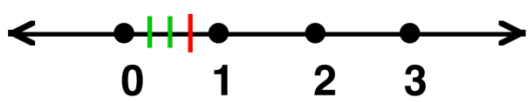
c) 

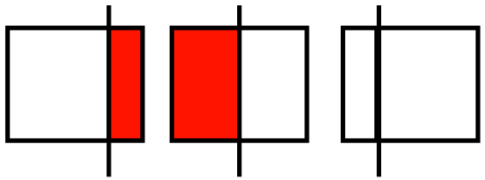
d) How many 4's are there in 3?

e) 18 crayons out of a box of 24

f) .75

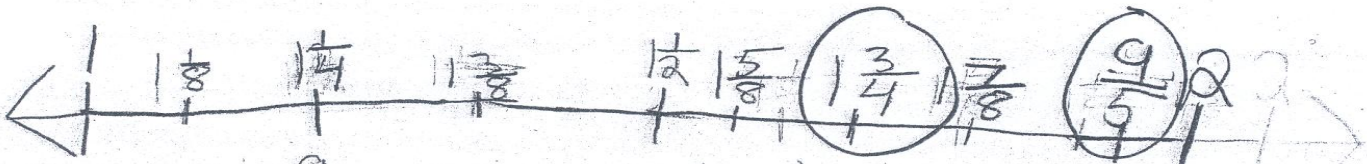
g) I want to share 3 bottles of soda equally among 4 people. How much will each person get?

h) 

i) 

1) Draw a number line and show where to place the fraction 9/5. Explain your thinking.

$1\frac{75}{100}$ $1\frac{80}{100}$



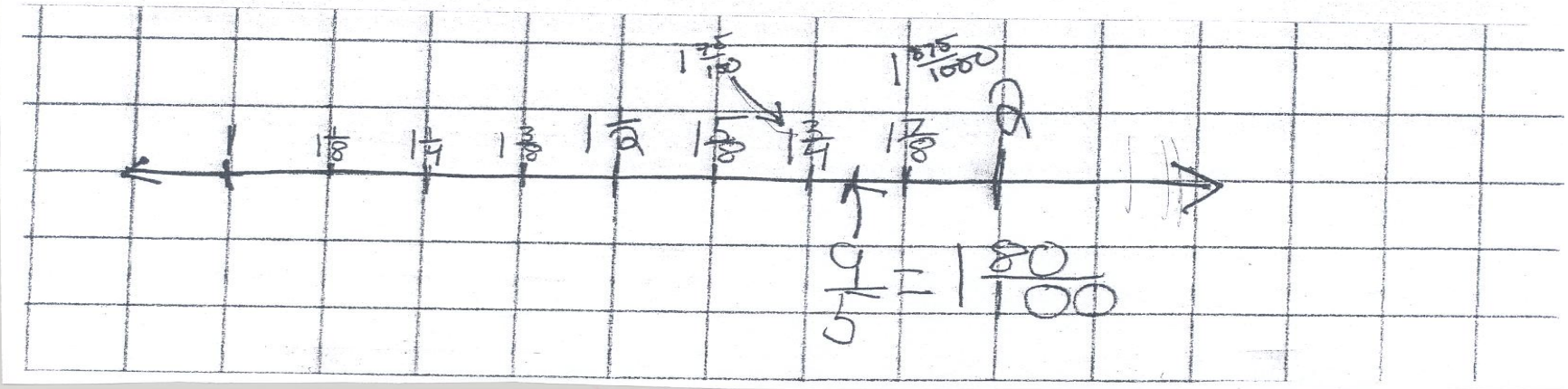
$\frac{9}{5}$ is equivlant to $1\frac{4}{5}$ and is almost 2 so it has to go there.

$\frac{7}{8} = \frac{87.5}{100}$

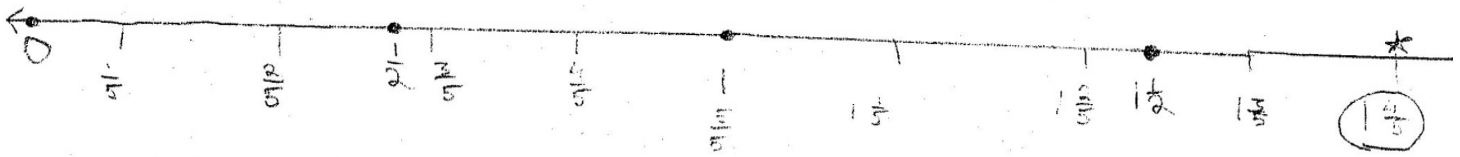
2) Order from smallest to greatest: 7/8, 3/8, 5/8, and 9/8.

$\frac{3}{8}$ $\frac{5}{8}$ $\frac{7}{8}$ $\frac{9}{8}$

$\frac{1}{8} = 12.5$



1) Draw a number line and show where to place the fraction 9/5.
Explain your thinking.



Because 9/5 is a topheavy fraction I said it was = to 1 4/5 ($\frac{9}{5}$, 9-5=$\frac{4}{5}$ 1 4/5), 1 4/5 is right behind 2 on the # line.

2) Order from smallest to greatest: 7/8, 3/8, 5/8, and 9/8.

$\frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{9}{8}$

3) Order from smallest to greatest: 3/5, 3/7, 3/4, and 3/8.

$\frac{3}{8}, \frac{3}{7}, \frac{3}{5}, \frac{3}{4}$

Fraction Sorting

- Sort the fractions below as near: 0, ½, or 1

4/7	1/7	8/9	3/5
2/3	1/10	4/8	6/11
4/5	2/12	9/12	5/12
1/8	3/8	4/9	7/14

- What's alike about all fractions near 1? Near 0?

True or False – 5/6

- More than ½?
- Less than 0.75?
- More than 80%?
- The ratio of boys to girls in our class?
- Between ½ and 1?
- Between ¾ and 1?
- Less than 1

$\frac{3}{4}$

- What happens to the value of the fraction if the numerator is increased by 1?
- What happens to the value of the fraction if the denominator is decreased by 1?
- What happens to the value of the fraction if the denominator is increased?

Ordering Fractions

Write these fractions in order from least to greatest. Tell how you decided.

- 5/3 5/6 5/5 5/4 5/8
- 7/8 2/8 10/8 3/8 1/8

- A student said that 3/4 and 5/6 are the same size because they both have one part missing: 3 is one less than 4 and 5 is one less than 6.

- Agree? Why or why not? How can you show the difference?

Decimals

Order the following:

7.1
0.71
0.711
71
71.1
711
71.11

What helps you in ordering the decimals?

You can’t make this stuff up!

- The weather reporter on WCRB (a Boston radio station) said there was a 30% chance of rain. The host of the show asked what that meant. The weather reporter said ``It will rain on 30% of the state." ``What are the chances of getting wet if you are in that 30% of the state?" ``100%.“

Recommendation 3

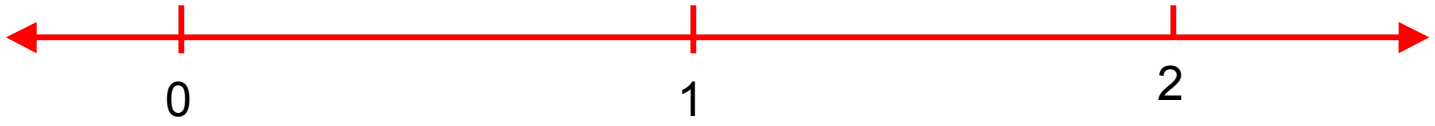
Help students understand why procedures for computations with fractions makes sense.

- Use area models, number lines, and other visual representations to improve students' understanding of formal computational procedures.
- *Provide opportunities for students to use estimation to predict or judge the reasonableness of answers to problems involving computation with fractions.*
- *Address common misconceptions regarding computational procedures with fractions.*
- Present real-world contexts with plausible numbers for problems that involve computing with fractions.

More than or less than ONE

- 1/12 + 2/3
- 5/6 + 1/3
- ½ + ¼
- 1 ½ - 7/8
- 2/3 x 2/3
- 5/6 x 7/8
- 4/5 ÷ 2/3
- 9/10 – 1/12

- Tell me about where 2/3 + 1/6 would be on this number line (Cramer, Henry, 2002).



Sense Making:

“2/3 is almost 1, 1/6 is a bit more, but the sum is < 1”

7/8 – 1/8 = ?

- Interviewer: Melanie these two circles represent pies that were each cut into eight pieces for a party. This pie on the left had seven pieces eaten from it. How much pie is left there?
- **Melanie:** ***One-eighth, writes 1/8*****.**
- Interviewer: The pie on the right had three pieces eaten from it. How much is left of that pie?
- **Melanie:** ***Five-eighths, writes 5/8.***
- Interviewer: If you put those two together, how much of a pie is left?
- **Melanie:** ***Six-eighths, writes 6/8.***
- Interviewer: Could you write a number sentence to show what you just did?
- **Melanie:** ***Writes 1/8 + 5/8 = 6/16.***
- Interviewer: That's not the same as you told me before. Is that OK?
- **Melanie:** ***Yes, this is the answer you get when you add fractions.***

What Happens Here?

- 1/2 x 3/4 **< or >** 3/4
- 3/4 x 1/2 **< or >** 1/2

- 1/2 ÷ 3/4 **< or >** 1/2
- 3/4 ÷ 1/2 **< or >** 3/4

Now what?

- There are 25 students in our class. Each student will get ¼ of a pizza. Your job is to find out how many pizzas we should order. Be sure to show your work.

- How many pizzas should we order?

- How might you represent 7 x 2/3 and would you think of 2/3 x 7 differently?

- If you shared 7 doughnuts among 3 people, how could you use this to help determine 2/3 x 7?

At Issue - Decimals

- Why start so early?
- Understanding of decimals draws on the understanding of fractions.
- 0.8 x 0.9 – think about 8/10 x 9/10

Recommendation 4

Develop students' understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.

- Develop students' understanding of proportional relations before teaching computational procedures that are conceptually difficult to understand (e.g., cross-multiplication). Build on students' developing strategies for solving ratio, rate, and proportion problems.
- Encourage students to use visual representations to solve ratio, rate, and proportion problems.
- Provide opportunities for students to use and discuss alternative strategies for solving ratio, rate, and proportion problems.

Lakers vs Nuggets

- Which player from the Lakers had the best shooting percentage
- Which player from the Lakers had the worst shooting percentage
- Same items for Nuggets
- Which players scored the most points, etc.

5th grade Kara's class

- I think my first not completely trivial mathematical observation was that the effect on a player's overall batting average from a single day's performance was much greater in the beginning of the season than at the
end.

- It's early I am hitting .200, 2 out of 10; I go 3 for 5 and I am now at 5 out of 15 and jump up to .333 an increase of over 100 points to the old BA! But if it's the end of the year and I am still at .200 hitting 100 out of 500 and go 3 for 5, now I am 103 out of 505 and only raise my average to .204. Still a big day, but too many bad days in between!

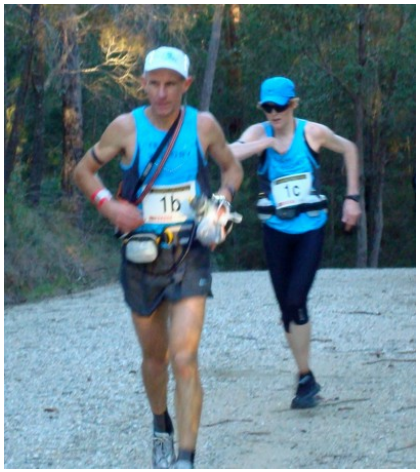
Ed Dubinsky, who has no idea how much I enjoyed this.

You can't make this stuff up

- Gettysburg Outlets – July 3, 2009. 50% off sale on all purchases at the Izod store. Sign indicates 50% off the all-store sale.
  - Patron – "well that means it's free."
  - Clerk – "no sir, it's 50% off the 50% off sale."
  - Patron – "well, 50% + 50% is 100% so that means it should be free."
  - This went on for a while. AND, there was a sign indicating 70% off for some items, meaning 70% off the 50% off original sale, which our patron would interpret as the item being free and 20% in cash!

For your next Ultra…

Trail Mix (for 6)



- ½ cup raisins
- ¾ cup peanuts
- 2/3 cup granola
- ½ cup dried fruit
- 2 tablespoons sunflower seeds
- ¼ cup M&Ms

Mix for the whole club – 30 runners
Make that 40 runners…

Adaptation of Bresser, Holtzman, 1999)

- On a scale 1” = 12 miles. If two places are 4” apart, how far are they away from each other in miles?

1”	12 miles
4”	

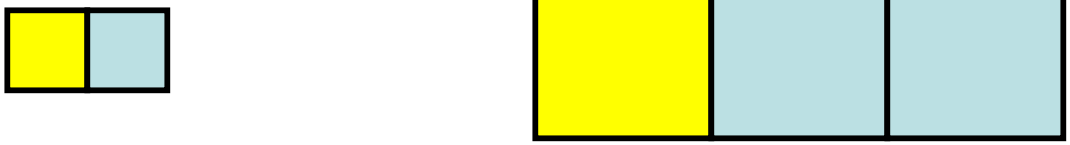
Recommendation 5

Professional development programs should place a high priority on improving teachers' understanding of fractions and of how to teach them.

- Build teachers' depth of understanding of fractions and computational procedures involving fractions.
- Prepare teachers to use varied pictorial and concrete representations of fractions and fraction operations.
- Develop teachers' ability to assess students' understandings and misunderstandings of fractions.

Fraction beginnings…

- Which one is larger, 1/2 or 1/3?



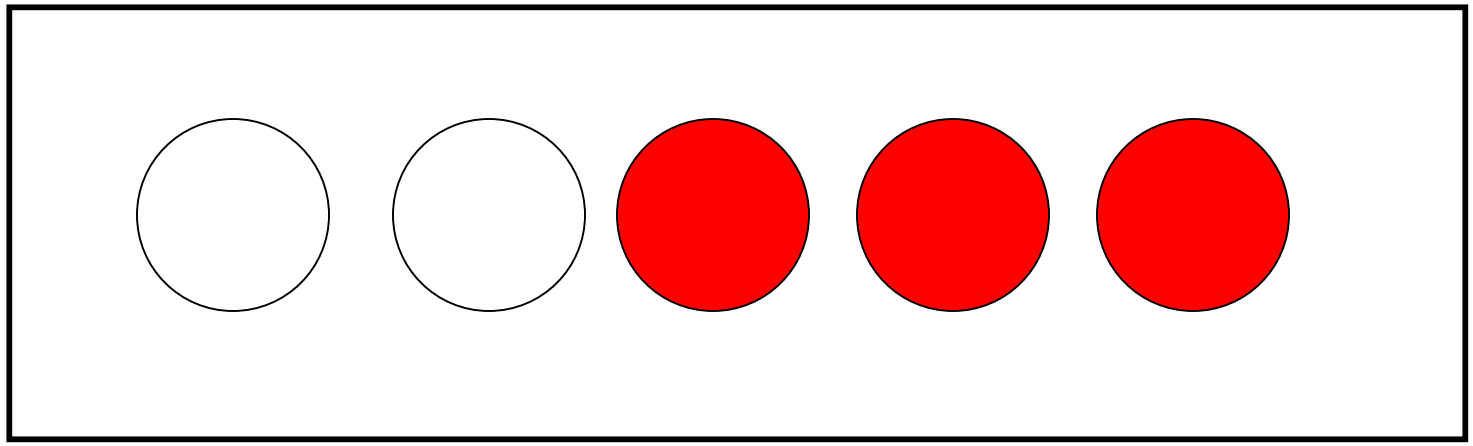
"the size of the fractional part is relative to the size of the whole…" (NCTM, 2006

Sharing and Dividing

- 5 want to share 11
- 5 want to share 4
- 10 want to share 6

Thinking about…

- ½ x ¼ =
- ½ ÷ ¼ =



Could the drawing above represent:

a. 5/3 of something
b. 3/5 of something
c. 1 ÷ 3/5 = 1 2/3
d. 5 ÷ 3 = 1 2/3

Thompson and Saldanha, NCTM, 2003, page 107

Concluding Thoughts

Recommendations

- Sharing and partitioning…;
- Fractions extend the number system (use this, CCSS);
- How procedures work and why;
- Applications – ratio, rate, and proportion
- Professional development needs – content and pedagogy

http://www.ffennell.com

Want the slides?