Coherence, Connections, Communication & Fraction Sense

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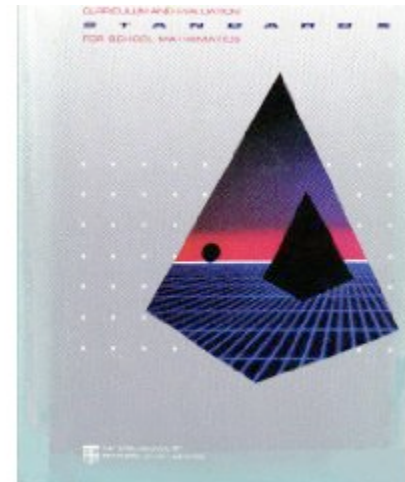*NCTM 2009 Annual Meeting & Exposition*
Washington, D.C.
April 24, 2009

All students should leave elementary and middle school with a strong sense of number

What does that mean?

How do we do that?

Some History!

Number Sense

- Number Meaning

- Relationships

- Magnitude

- Operation Sense

- Real Life Number Sense - Applications

Howden, 1989

Making Sense of Numbers…

1. Ability to compose and decompose numbers…
2. Ability to recognize the relative magnitude of numbers – including comparing and ordering.
3. Ability to deal with the absolute magnitude of numbers – realizing, for instance there are far fewer than 500 people in this session!
4. Ability to use benchmarks.
5. Ability to link numeration, operation, and relation symbols in meaningful ways.
6. Understanding the effects of operations on numbers.
7. The ability to perform mental computation through invented strategies that take advantages of numerical and operational properties.
8. Being able to use numbers flexibly to estimate numerical answers to computations, and to recognize when an estimate is appropriate.
9. A disposition towards making sense of numbers.

"It is possible to have good number sense for whole numbers, but not for fractions…"

Sowder, J. and Schappelle, Eds., 1989

And more recently…

- In its most fundamental form, number sense entails an ability to immediately identify the numerical value associated with small quantities;

- …this more highly developed form of number sense should extend to numbers written in fraction, decimal, and exponential forms.

- …poor number sense interferes with learning algorithms and number facts and prevents use of strategies to verify if solutions to problems are reasonable.

NMAP, page 27

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How did this get started…

Fraction issues…

- Conceptual Knowledge and Skills
- Learning Processes
- Assessment
- Survey of Algebra Teachers

Foundations for Success

National Mathematics Advisory Panel

FINAL REPORT · SPRING 2008
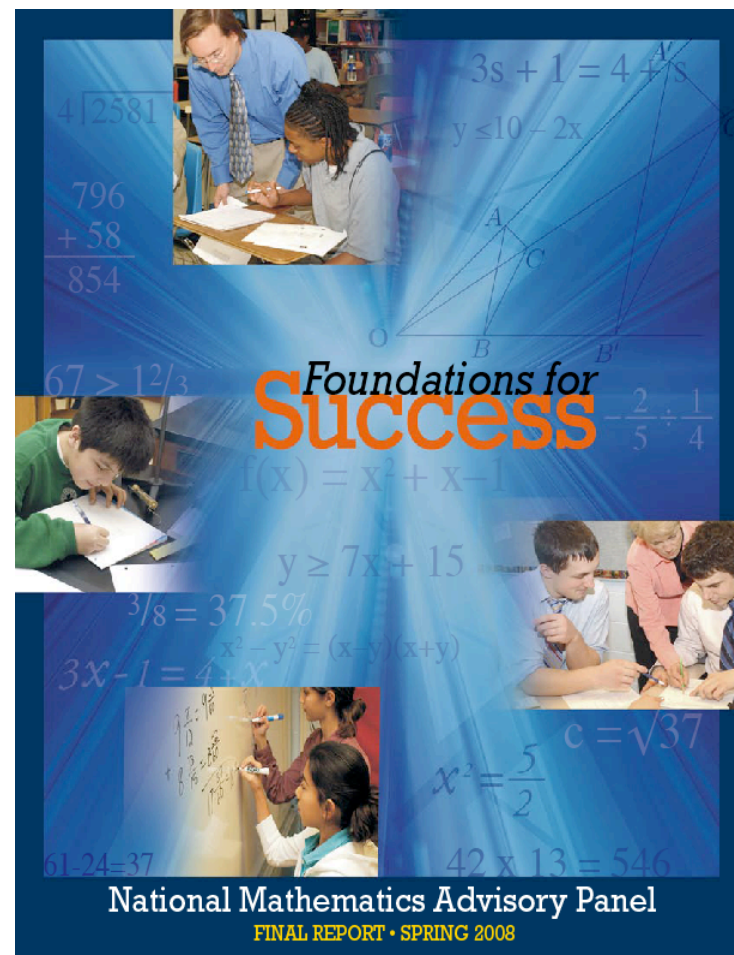
NMAP - Student Preparation

- The first question concerned the adequacy of student preparation coming into the Algebra I classes. The topics that were rated as especially problematic were:
    - Rational numbers;
    - Solving word problems, and;
    - Basic study skills.

Final Report on the National Survey of Algebra Teachers for the National Math Panel, NORC, September, 2007

Do you have a sense of number?

- Is 7/8 > 5/6? How do you know?

- What is 6.2 x 12?

- If the restaurant bill was $119.23, how much of a tip should you leave?

- If a 10-year old is 5' tall, how tall will the child be at age 20?

- Each MLB team plays is 162 games. Suggest a wins to losses ratio for the Washington Nationals.
- At this writing the Nationals had a record of 1 win and 7 losses.

Policy and Political Issues

- Number sense is developed!
- Where <u>and how</u> does this fit in any state's curricular standards?
- This is more than whole numbers!

Curriculum Issues

Grade 1	Grade 3
• Equal Parts • Halves – region model • Thirds, Fourths – region model • Fractions of a set – halves, thirds, fourths • Non-unit fractions – Region and set models	• Dividing Regions into Equal Parts • Identifying Fractions – region model • Identifying Fractions – set model • Fraction Benchmarks • Equivalent Fractions – fraction bars • Comparing Fractions – fraction bars • Fractions on the Number Line • Adding Fractions – Like denominators – fraction bars • Subtracting Fractions - Like denominators – fraction bars • Fractions and Decimals • Money and Decimals • Adding and Subtracting Money

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Curriculum Issues

Grade 5

- Identifying Fractions – region, set and number line models
- Fractions and sharing – multiple models
- Mixed Numbers and Improper Fractions
- Decimals – tenths and hundredths
- Decimals – thousandths
- Comparing and Ordering Fractions and Decimals on the Number Line.
- Equivalent Fractions – region model
- Equivalent Fractions – number line
- Simplest Form
- Comparing Fractions – fraction bars
- Comparing Fractions – denominators, numerators
- Ordering Fractions
- Comparing and Ordering Mixed Numbers

Grade 5 (cont.)

- Adding Fractions – Like denominators – fraction bars
- Subtracting Fractions - Like denominators – fraction bars
- Adding and Subtracting Fractions – Number Line
- Adding and Subtracting – Unlike Denominators
- **Applications in various topic areas**

Curriculum Issues

Grade 7

- Adding and Subtracting Decimals
- Estimating Decimal Sums and Differences
- Multiplying Decimals – area model
- Dividing Decimals – tenth strips
- Simplifying Fractions
- Comparing and Ordering Fractions
- Mixed Numbers and Improper Fractions
- Fractions and Decimals – convert
- Comparing Rationals – Number Line
- Estimating with Fractions and Mixed Numbers – all operations
- Add and Subtract Fractions and Mixed Numbers
- Multiplying Fractions and Mixed Numbers
- Dividing Fractions and Mixed Numbers

Grade 7 (cont.)

- Ratios
- Unit Rates
- Proportions
- Similar Figures and Proportion
- Maps and Scale Drawings and Proportion
- Percents – Models (grid)
- Ratio and Percent
- Percents, Fractions, and Decimals
- Percents > 100% and < 1%
- Percent of a Number
- Percent and Proportion
- Percent and Equations
- Applications of Percent
- Percent of Change
- **Applications throughout – all other topics**

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What about research?

"…has a terrible time with basic skills. I mean, if we ever do anything with fractions, she's lost."*

*a lament by and about far too many…

Usher, …Middle School Students' Self-Efficacy in Mathematics, AERJ, 2009

2nd Handbook Research on Mathematics Teaching & Learning

- Whole Number Concepts and Operations
  - Citations: 334
- Rational Numbers and Proportional Reasoning
  - Citations: 140
  - 2000's: 9 citations;
    - 109 in Whole Number Concepts and Operations

NCTM, 2007

- "The number of references in this chapter predating 1992 is far greater than the number appearing since the last handbook."
- "This crisis…stems from:
  - Teachers are not prepared to teach content other than part-whole fractions;
  - Long-term commitment is needed because rational number topics are learned over many years.
  - The nonlinear development of the content does not mesh well with scope and sequence currently prescribing mathematics instruction in schools; and
  - In comparison to a domain such as early addition and subtraction, little research progress is evident."

Lamon, 2007 in 2nd Handbook Research on Mathematics Teaching & Learning

JRME and Fractions…

- March 2005
  - Representing Fractions with Standard Notation: A Developmental Analysis. Geoffrey B. Saxe, Edd V. Taylor, Clifton McIntosh and Maryl Gearhart
- January 2008
  - Teaching and Learning Fraction Addition on Number Lines: Andrew Izsák, Erik Tillema and Zelha Tunç-Pekkan
- May 2008
  - Learning to Use Fractions: Examining Middle School Students' Emerging Fraction Literacy: Debra I. Johanning
- July 2008
  - Josh's Operational Conjectures: Abductions of a Splitting Operation and the Construction of New Fractional Schemes: Anderson Norton
- March 2009
  - Students' Overuse of Proportionality on Missing-Value Problems: How Numbers May Change Solutions: Wim Van Dooren, Dirk De Bock, Marleen Evers and Lieven Verschaffel

5 manuscripts in approx 4+ years

McDANIEL COLLEGE

gismo
Generating Increased Science & Math Opportunities

gismo
Rational Number Reasoning Database

Funded by the National Science Foundation

Confrey and Mahoney, 2009, North Carolina State University

Another look…

4. What fraction of the group of umbrellas is closed?

A) $\frac{1}{3}$

B) $\frac{3}{7}$

C) $\frac{4}{7}$

D) $\frac{3}{4}$

Correct 80%
Incorrect 19%
Omit 1%

NAEP, 2007

Grade 4

What fraction of the figure is shaded?
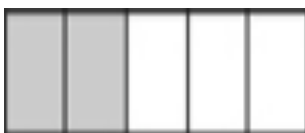
Correct: 77%
Incorrect: 22%
Omit: 1%

Answer: _____

Did you use the calculator on this question?

NAEP, 2007

McDANIEL
C O L L E G E

Luis wants to make a game spinner in which the chance of landing on blue will be twice the chance of landing on red. He is going to label each section either red (R) or blue (B). Show how he could label his spinner.
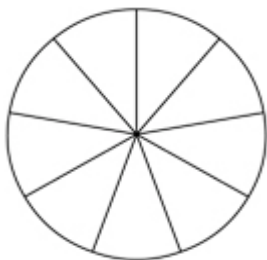
11% satisfactory
8% extended
14% partial
4% minimal
59% incorrect

Number of blues _____

Number of reds _____

NAEP, 2007

In which of the following are the three fractions arranged from least to greatest?

A) $\frac{2}{7}, \frac{1}{2}, \frac{5}{9}$

B) $\frac{1}{2}, \frac{2}{7}, \frac{5}{9}$

C) $\frac{1}{2}, \frac{5}{9}, \frac{2}{7}$

D) $\frac{5}{9}, \frac{1}{2}, \frac{2}{7}$

E) $\frac{5}{9}, \frac{2}{7}, \frac{1}{2}$

49% correct
49% incorrect
1% omit

O Yes   O No

Did you use the calculator on this question?

NAEP, 2007

Grade 8

Tammy scored 52 out of 57 possible points on a quiz. Which of the following is closest to the percent of the total number of points that Tammy scored?

A) 0.91%
B) 1.10%
C) 52%
D) 91%
E) 95%

62% correct
37% incorrect
1% omit

Did you use the calculator on this question?

NAEP, 2007

Grade 8

The temperature in degrees Celsius can be found by subtracting 32 from the temperature in degrees Fahrenheit and multiplying the result by 5/9. If the temperature of a furnace is 393 degrees Fahrenheit, what is it in degrees Celsius, to the nearest degree?

A) 649
B) 375
C) 219
D) 201
E) 187

35% correct
61% incorrect
3% omit

Did you use the calculator on this question?

NAEP, 2007

- Fractions are a major area of study in upper elementary school mathematics. It is time to shift the emphasis and redefine the goal of fraction instruction from learning computation rules to developing fraction operation sense (Huinker, 2002).

- Do we do this?

Number Meaning - Critical Issues

Whole Numbers:

- Number Meaning

- Counting - Counting on, Counting back

- Composing and Decomposing

And for fractions…

K level – sharing, halves, smaller halves, still smaller halves
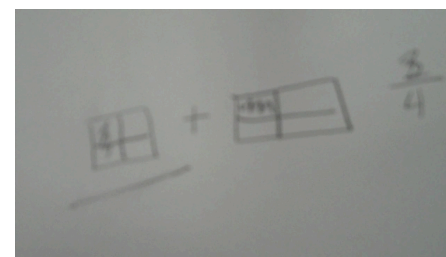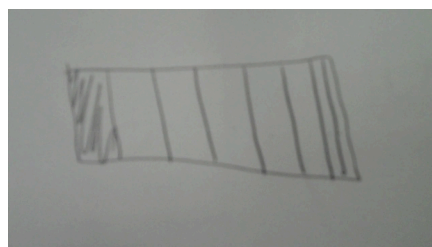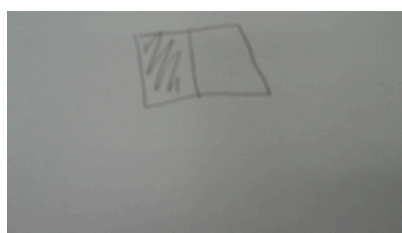
Grade 4 level – half, one eighth, ½ + ¼
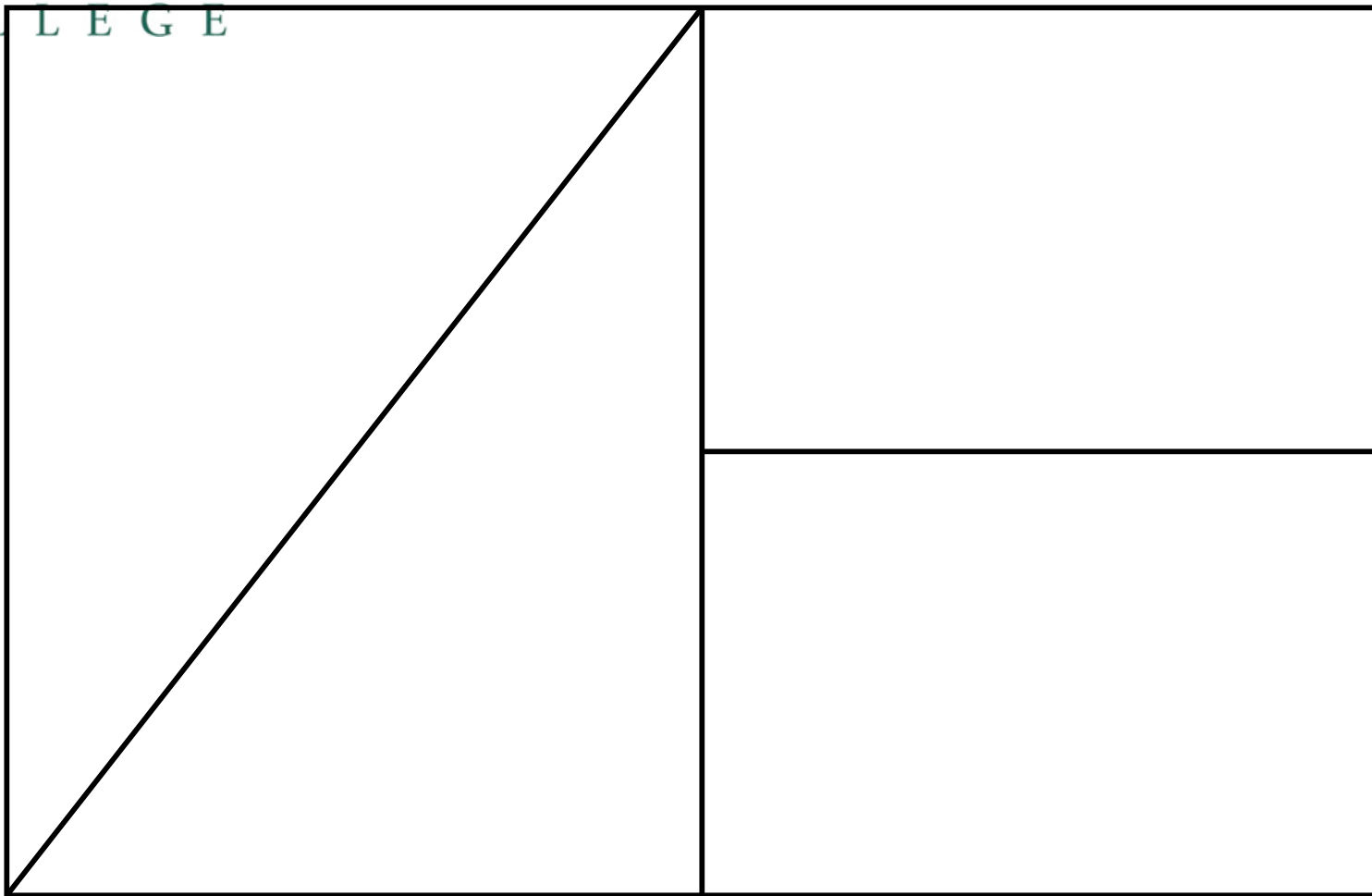
When the cake is cut, size of each piece?

Fraction beginnings…

- Which one is larger, 1/2 or 1/3?
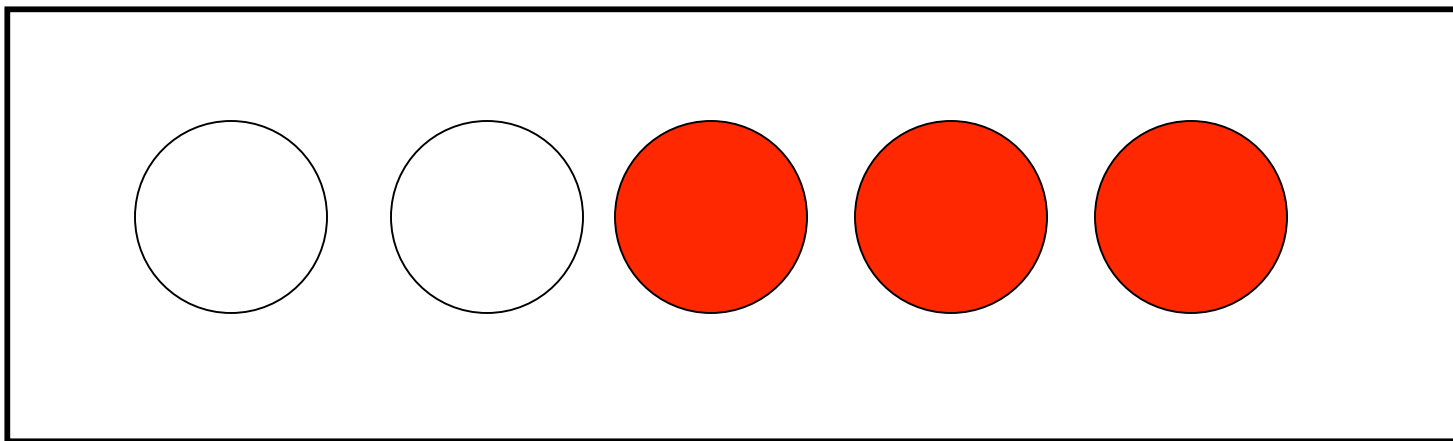
Could the drawing above represent:

a. 5/3 of something
b. 3/5 of something
c. 1 ÷ 3/5 = 1 2/3
d. 5 ÷ 3 = 1 2/3

- Name
  - A fraction between 0 and 1.
  - A fraction between ¼ and ½
  - A fraction between 1/3 and 2/3
  - A fraction between 5/6 and 1
  - A fraction between 0 and 1/8
  - A decimal between 1.1 and 1.2
  - A decimal between 1.10 and 1.11
  - …

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Which is more?

$\frac{5}{6}$ or $\frac{5}{12}$ of a chocolate candy bar?

Which is more?

$\frac{5}{6}$ or $\frac{5}{12}$ of a chocolate candy bar?

How do you know?

How do you know?

$\frac{5}{6}$ is larger than $\frac{5}{12}$

Mirra, Grade 3-5 Focal Points, NCTM, 2008

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Which is more?
$\frac{1}{6}$ or $\frac{2}{3}$ of a pizza?

How do you know?

$\frac{1 \times 1}{6 \times 1}$ $\frac{2 \times 2}{\times 3}$ $\frac{4}{6}$ $\frac{2}{3} = \frac{4}{6}$

$\downarrow \frac{1}{6}$ $\downarrow \frac{2}{3}$
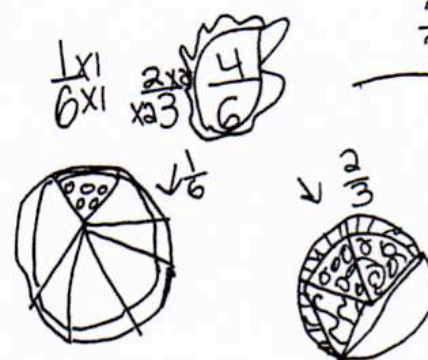
3 quarters of the pizza is better than

Which is more?
$\frac{1}{6}$ or $\frac{2}{3}$ of a pizza?

How do you know?

$\rightarrow \frac{2}{3}$ because...

$\frac{1}{6}$ of a pizza
6
5 2
4 3
is only 1 slice

$\frac{2}{3}$ of a pizza
3 is almost whole

Mirra, Grade 3-5
Focal Points, NCTM,
2008

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Math Wall Activities

2%

100%

3/4

3.11

Introducing fractions using sharing

- Four children want to share 10 cookies. Each child gets the same amount. How many cookies does each child get?

- Why?
  - Sharing is intuitive
  - Solution combines wholes and fractions (2½)
  - Sharing and repeated halving

Fraction Sorting

- Sort the fractions below as near: 0, ½, or 1

4/7	1/7	8/9	3/5
2/3	1/10	4/8	6/11
4/5	2/12	9/12	5/12
1/8	3/8	4/9	7/14

- What's alike about all fractions near 1? Near 0?

Show two equivalent fractions for 2/8
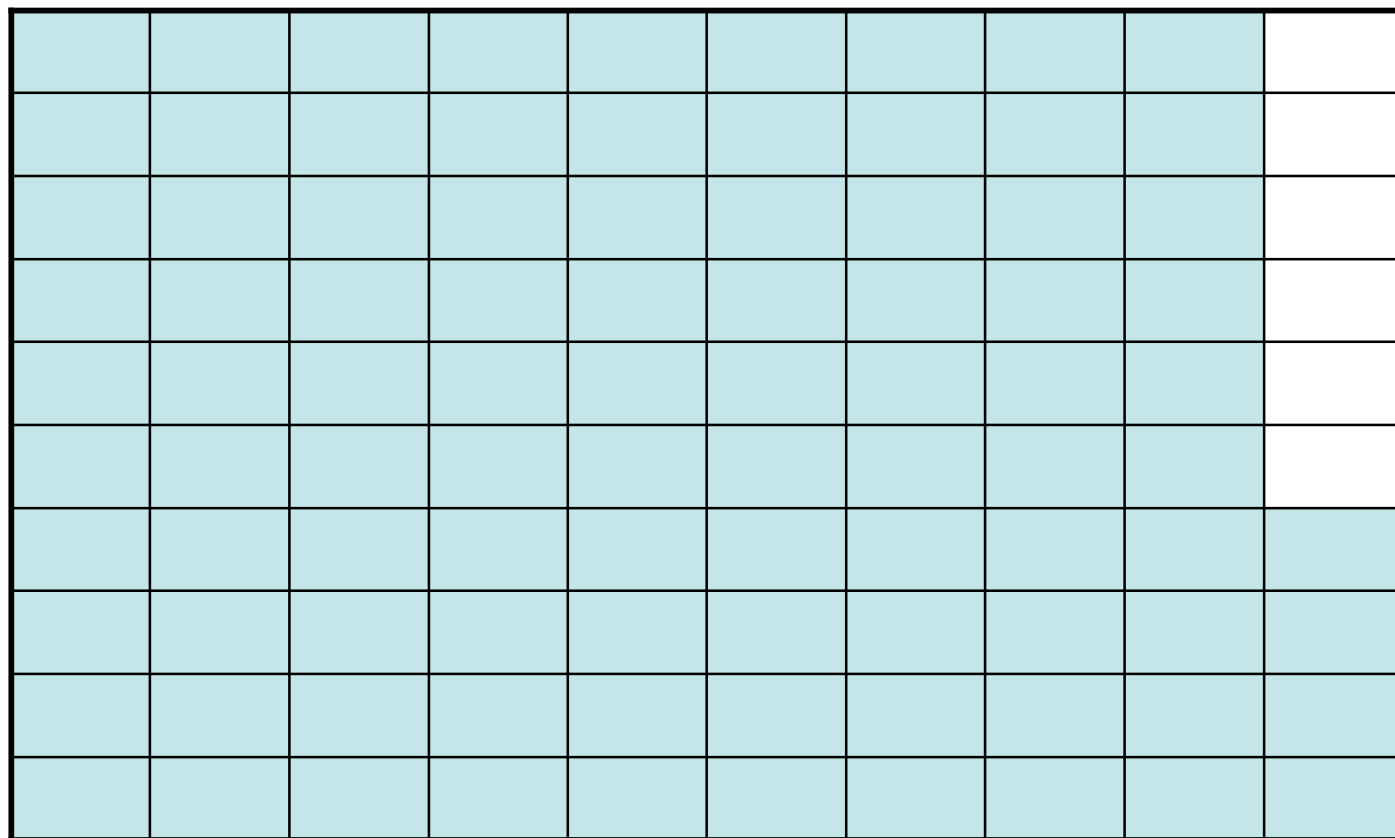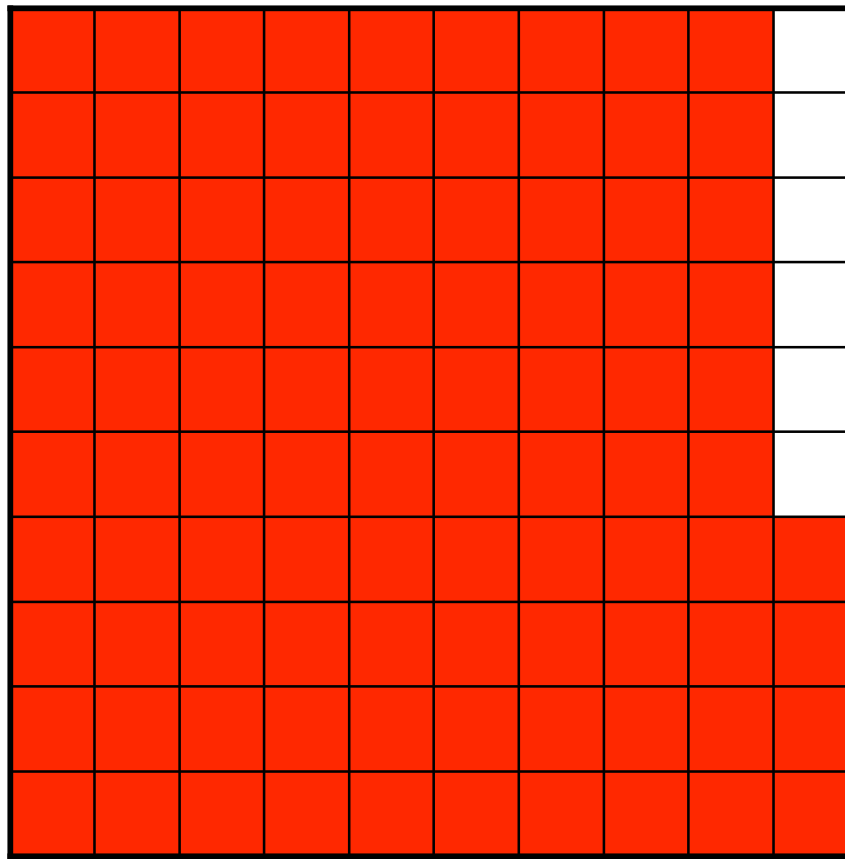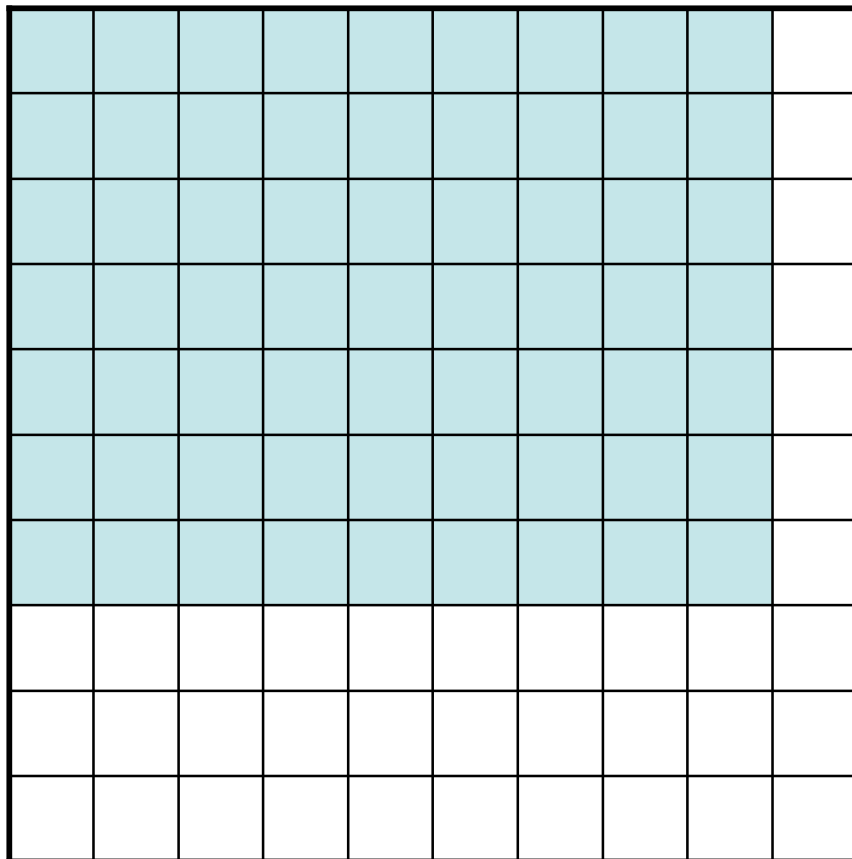
Close to 1? How much?

How much?

Sense Making

- How can we share **eleven** hoagies (aka subs) among four people?

- How can we share **eleven** hoagies (aka subs) among five people?

Ballparking

- About, around, close to:

  ½

  ¼

  25%

  10%

  1/3

  2/3

  75%

  200%

1. **Provide fraction or % and have students provide one close to that amount.**

2. **Provide a context and have them write about that amount/context using one of the ballpark amounts.**

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True or False – 5/6

- More than ½?
- Less than 0.75?
- More than 80%?
- The ratio of boys to girls in our class?
- Between ½ and 1?
- Between ¾ and 1?
- Less than 1

- What happens to the value of the fraction if the numerator is increased by 1?

- What happens to the value of the fraction if the denominator is decreased by 1?

- What happens to the value of the fraction if the denominator is increased?
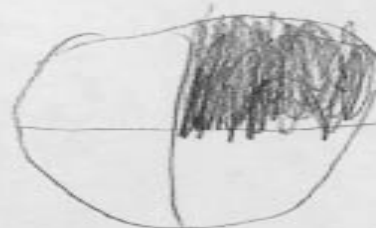
Ordering Fractions

Write these fractions in order from least to greatest. Tell how you decided.

- 5/3   5/6   5/5   5/4   5/8

- 7/8   2/8   10/8   3/8   1/8

Name Zachary collins   Grade level 5
School Deep Run Elemetry school

**Explain how you know that 1/3 is greater than 1/4. Use
pictures, numbers or words in your explanation.**

$$\frac{1}{3} \qquad \frac{4}{12}$$

$$\frac{1}{4} \qquad \frac{3}{12}$$

1/3 is greater than 1/4
because when you do the
magic one trike it is greater
by 1/12 more.

- Tell me about where 2/3 + 1/6 would be on this number line (Cramer, Henry, 2002).

```
<-----|---------------|---------------|----->
      0               1               2
```
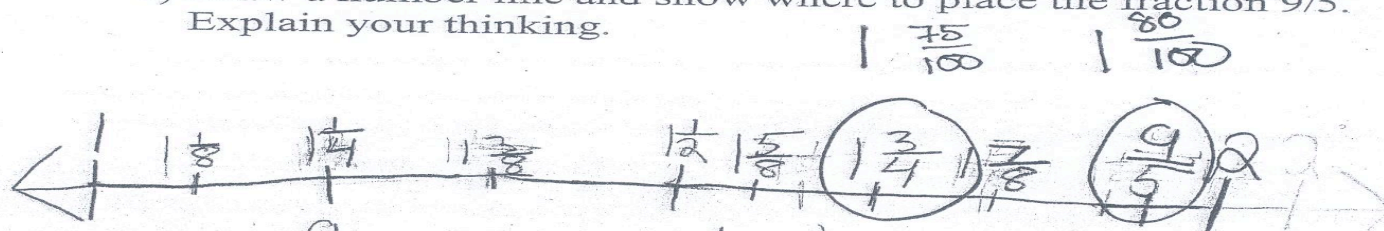
Sense Making:

"2/3 is almost 1, 1/6 is a bit more, but the sum is < 1"

1) Draw a number line and show where to place the fraction 9/5.
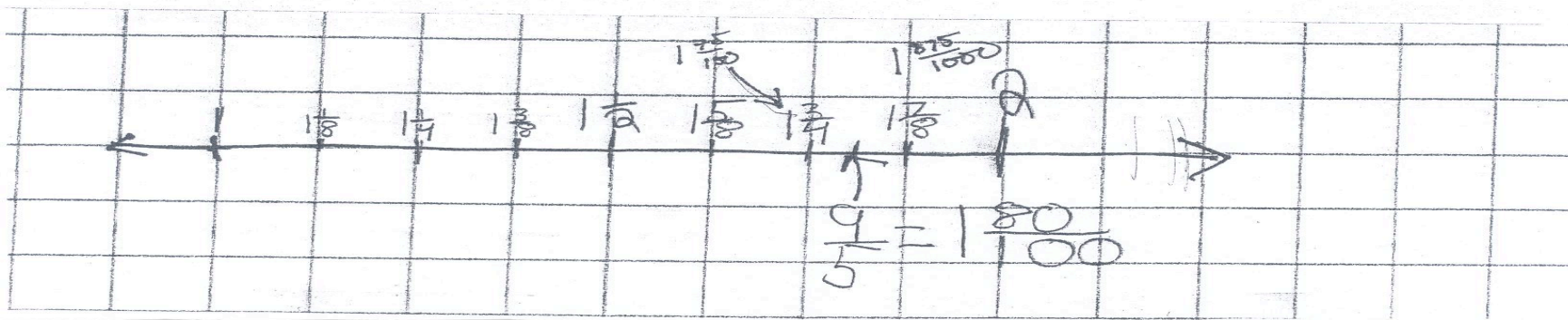   Explain your thinking.

$1\frac{75}{100}$ $1\frac{80}{100}$

$\frac{9}{5}$ is equivalent to $1\frac{4}{5}$ and is almost 2 so it has to go there.

$\frac{7}{8} = \frac{87.5}{100}$

2) Order from smallest to greatest: 7/8, 3/8, 5/8, and 9/8.

$\frac{3}{8}$ $\frac{5}{8}$ $\frac{7}{8}$ $\frac{9}{8}$

$\frac{1}{8} = 12.5$

$\frac{9}{5} = 1\frac{80}{100}$
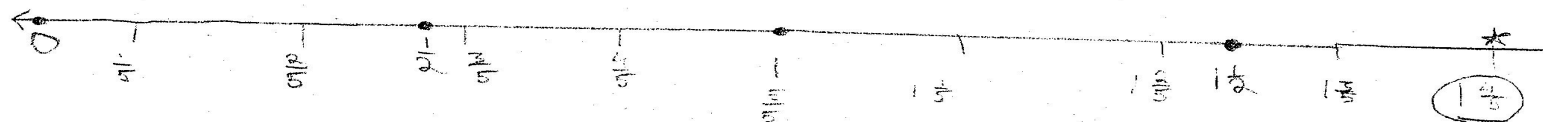
1) Draw a number line and show where to place the fraction 9/5. Explain your thinking.

Because 9/5 is a topneovy fraction I said it was = to 1⁴/₅ ($\frac{9}{5}$, 9-5=$\frac{4}{5}$ 1⁴/₅), 1⁴/₅ is right behind 2 on the # line.

2) Order from smallest to greatest: 7/8, 3/8, 5/8, and 9/8.

$$\frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{9}{8}$$

3) Order from smallest to greatest: 3/5, 3/7, 3/4, and 3/8.

$$\frac{3}{8}, \frac{3}{7}, \frac{3}{5}, \frac{3}{4}$$

³⁄₄

What happens to the value of the fraction if the numerator AND denominator are doubled?

- A student said that 3/4 and 5/6 are the same size because they both have one part missing – 3 is one less than 4 and 5 is one less than 6.

- Agree? Why or why not? How can you show the difference?

Missing Numbers

- What's my number?
- $2x + 7 = y$
- Rule: Double the number and add 7. What's the number if x =

$$10$$

$$100$$

$$0.1$$

$$0.01$$

Use Percent – Don't Wait!

- Put 2/3; 0.5 and 3/4 in order from smallest to largest.

- It's easy, 0.5 is 50% and 2/3 is 66%, and so it goes first 0.5, then 2/3 and then ¾ because that's 75%.*

*response by Andy in New Approaches to Teaching the Rational Number System by Joan Moss in How Students Learn: Mathematics in the Classroom, NRC, 2005.

Operations and Sense Making

More than or less than ONE

- 1/12 + 2/3

- 5/6 + 1/3

- ½ + ¼

- 1 ½ - 7/8

- 2/3 x 2/3

- 5/6 x 7/8

- 4/5 ÷ 2/3

- 9/10 – 1/12

- Estimate the answers

- ½ + 1/3
- 2/3 + 5/6
- 7/8 + 1/3
- 1 4/5 + ¾
- 7 ½ - 4.8

Use number lines to show estimated placement of sums and differences, etc.

What Happens Here?

- 1/2 x 3/4   < or >   3/4
- 3/4 x 1/2   < or >   1/2

- 1/2 ÷ 3/4   < or >   1/2
- 3/4 ÷ 1/2   < or >   3/4

The quotient is 0.5, what is the divisor and dividend?

Boxes to multiply…

- Draw a rectangle to show 4 ½ x 7 ¼ =

	7	¼
4	28	1
½	3 ½	1/8

$$4 \frac{1}{2} \times 7 \frac{1}{4} = (4 \times 7) + (\frac{1}{2} \times 7) + (4 \times \frac{1}{4}) + (1/2 \times 1/8)$$

Navigations 3-5, Number and Operations, 2007, sorta

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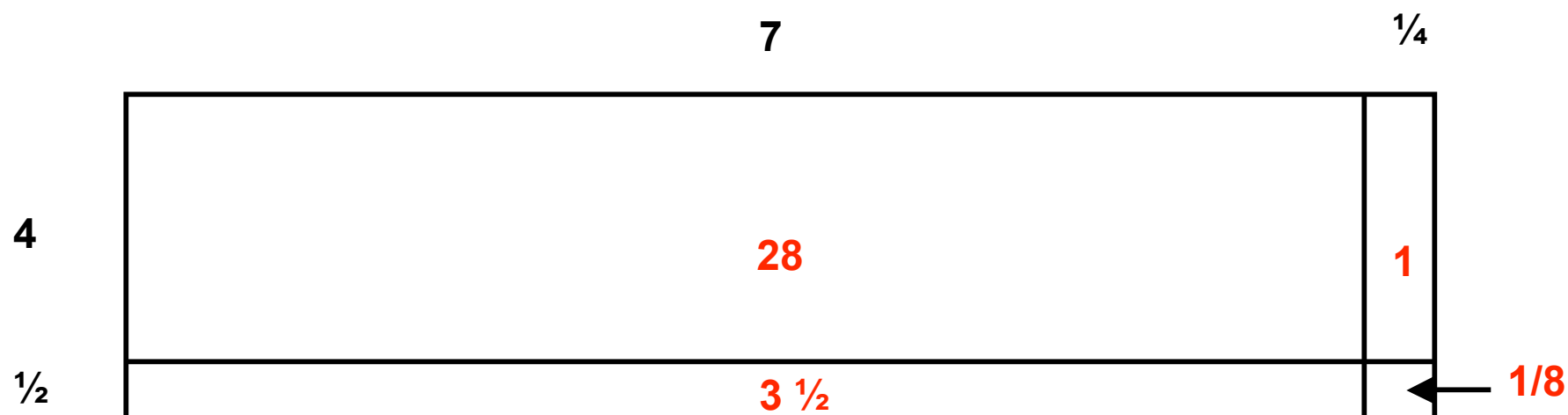
- How might you represent 7 x 2/3 and would you think of 2/3 x 7 differently?

- If you shared 7 doughnuts among 3 people, how could you use this to help determine 2/3 x 7?

Decimals - What Happens?

Number	x 0.05	x 0.48	x 0.9
100			
60			
24			
?			

- In general, what happens when you multiply a whole number by: 0.05; 0.48; 0.9?
- Begin thinking of 0.05 as 5% or nickel:dollar, etc.

Mental Math

- (25 x 16) x ¼

- 3 1/8 x 24

- 4/5 x 15

- 68 x 0.5
- 0.25 x 48
- 0.2 x 375
- 0.05 x 280
- 56 x 0.125
- 0.75 x 72

Mental Math

- 25% x 44
- 75% x 24
- 50% x 76
- 33 1/3% x 93
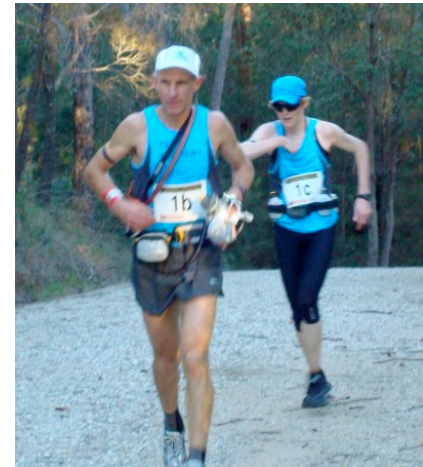- 38% x 50
- 84% x 25

Context is good

Favorites

- Write **3** numbers that have some significance to your life – these must be fractions, decimals, percents.

- Exchange lists. Provide random clues for the numbers.

- Guess which numbers fit the clues.

For your next Ultra…

Trail Mix (for 6)

- ½ cup raisins
- ¾ cup peanuts
- 2/3 cup granola
- ½ cup dried fruit
- 2 tablespoons sunflower seeds
- ¼ cup M&Ms

Mix for the whole club – 30 runners
Make that 40 runners…

Adaptation of Bresser, Holtzman, 1999)

- On a scale 1" = 12 miles. If two places are 4" apart, how far are they away from each other in miles?

1"	12 miles
4"	

Now what?

- There are 25 students in our class. Each student will get ¼ of a pizza. Your job is to find out how many pizzas we should order. Be sure to show your work.

- How many pizzas should we order?

Fractions!

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Percent Benchmarks

0%

100%   50%   < 10%

~25%   ~75%   ~90%

> 50%   < 50%

- Lefthanders in the room or class
- Once lived in **New Jersey**
- Been involved in education > 10 years
- People who were born in **Washington, D.C.**

Reys, et al, 1991

Rational Number Sense

- Using representations of fractions, decimals, and percents interchangeably;

- Comparing and ordering fractions, decimals and percents;

- Using benchmarks to estimate when comparing and ordering and in determining sums, differences, products, and quotients.

- (Adapted from Moss, 2002)

Concluding Thoughts

- Number sense is elusive
- Number sense should be nurtured – every day!
- A sense of number breeds confidence.
- Fractions – all of 'em - are numbers too!
- Decimals and inconsistency…
- Probability – now!

- Represent a solution for each of the following. Make sure to indicate how each sum, difference, product, or quotient is represented.

$$2\frac{1}{5} + \frac{1}{3} \qquad \frac{7}{8} - \frac{1}{2} \qquad \frac{1}{4} \, x \, \frac{2}{5}$$
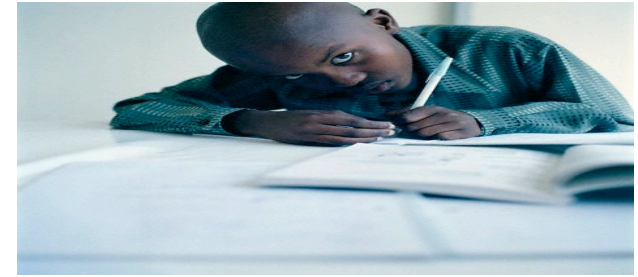
$$\frac{5}{6} \div \frac{1}{4} \qquad\qquad 2\frac{1}{2} \div \frac{1}{3}$$
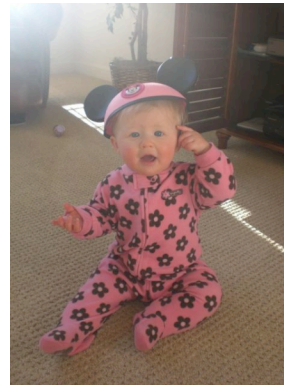
Plastics…

A major goal for K-8 mathematics
should be proficiency with fractions…

NMAP, 2008

We share a responsibility

This must be a concerted team effort, it's about every child every day.

MCDANIEL COLLEGE

It has been my pleasure to serve all of you!

Thank you!

www.ffennell.com

Go Hank!

Maryland Council of Teachers of Mathematics

Thanks Marylanders